4 Algebras, Operators and Dual Spaces

4.1 The Stone-Weierstraß Theorem

Definition 4.1. A vector space X over the field $\mathbb{K}$ is called an *algebra* over $\mathbb{K}$ iff it is equipped with an associative bilinear map $\cdot : X \times X \to X$. This map is called *multiplication*.

Definition 4.2. Let X be an algebra over $\mathbb{K}$ equipped with a topology. Then X is called a *topological algebra* iff vector addition, scalar multiplication and algebra multiplication are continuous.

Proposition 4.3. *Let S be a topological space. Then, $\mathrm{C}(S, \mathbb{K})$ with the topology of compact convergence is a topological algebra.*

Proof. **Exercise**. □

Lemma 4.4. *Let $c > 0$. The absolute value function $|\cdot| : \mathbb{R} \to \mathbb{R}$ given by $x \mapsto |x|$ can be approximated uniformly on $[-c, c]$ by polynomials with vanishing constant term.*

Proof. **Exercise**. □

Lemma 4.5. *Let $c > 0$ and $\epsilon > 0$. Then, there exist polynomials P_{min} and P_{max} of n variables and without constant term such that for all $a_1, \ldots, a_n \in [-c, c]$,*

$$|P_{min}(a_1, \ldots, a_n) - \min\{a_1, \ldots, a_n\}| < \epsilon,$$
$$|P_{max}(a_1, \ldots, a_n) - \max\{a_1, \ldots, a_n\}| < \epsilon.$$

Furthermore, $P_{min}(a, \ldots, a) = a$ and $P_{max}(a, \ldots, a) = a$.

Proof. It suffices to show the statement for $n = 2$. Since the minimum and maximum functions can be evaluated iteratively, the general statement follows then by iteration and a multi-ϵ argument. We notice that

$$\max\{a_1, a_2\} = \frac{a_1 + a_2}{2} + \frac{|a_1 - a_2|}{2}$$
$$\min\{a_1, a_2\} = \frac{a_1 + a_2}{2} - \frac{|a_1 - a_2|}{2}.$$

By Lemma 4.4 there exists a polynomial P without constant terms such that $|P(x) - |x|| < 2\epsilon$ for all $x \in [-2c, 2c]$. It is easily verified that

$$P_{\max}(a_1, a_2) := \frac{a_1 + a_2}{2} + \frac{P(a_1 - a_2)}{2},$$
$$P_{\min}(a_1, a_2) := \frac{a_1 + a_2}{2} - \frac{P(a_1 - a_2)}{2}$$

have the desired properties. □

Definition 4.6. Let S be a set and $A \subseteq F(S, \mathbb{K})$. We say that A *separates points* iff for each pair $x, y \in S$ such that $x \neq y$ there exists $f \in A$ such that $f(x) \neq f(y)$. We say that A vanishes nowhere iff for each $x \in S$ there exists $f \in A$ such that $f(x) \neq 0$.

Lemma 4.7. *Let S be a topological space and $A \subseteq \mathrm{C}(S, \mathbb{K})$ a subalgebra. Suppose that A separates points and vanishes nowhere. Then, for any pair $x, y \in S$ with $x \neq y$ and any pair $a, b \in \mathbb{K}$ there exists a function $f \in A$ such that $f(x) = a$ and $f(y) = b$.*

Proof. **Exercise.** □

Theorem 4.8 (real Stone-Weierstraß)**.** *Let K be a compact Hausdorff space and $A \subseteq \mathrm{C}(K, \mathbb{R})$ a subalgebra. Suppose that A separates points and vanishes nowhere. Then, A is dense in $\mathrm{C}(K, \mathbb{R})$ with respect to the topology of uniform convergence.*

Proof. Given $f \in \mathrm{C}(K, \mathbb{R})$, and $\epsilon > 0$ we have to show that there is $k \in A$ such that $k \in B_\epsilon(f)$, i.e.,

$$f(x) - \epsilon < k(x) < f(x) + \epsilon \quad \forall x \in K.$$

Fix $x \in K$. For each $y \in K$ we choose a function $g_{x,y} \in A$ such that $f(x) = g_{x,y}(x)$ and $f(y) = g_{x,y}(y)$. This is possible by Lemma 4.7. By continuity there exists an open neighborhood U_y for each $y \in K$ such that $g_{x,y}(z) < f(z) + \epsilon/4$ for all $z \in U_y$. Since K is compact there are finitely many points $y_1, \ldots, y_n \in K$ such that the associated open neighborhoods $U_{y_1}, \ldots, U_{y_n}$ cover K. Let

$$g_x := \min\{g_{x,y_1}, \ldots, g_{x,y_n}\}.$$

Since K is compact there exists $c > 0$ such that $|g_{x,y_i}(z)| \leq c$ for all $z \in K$ and all $i \in \{1, \ldots, n\}$. Then, by Lemma 4.5 there exists a polynomial $P_{\min}$ such that $h_x := P_{\min}(g_{x,y_1}, \ldots, g_{x,y_n}) \in A$ satisfies $|h_x(z) - g_x(z)| < \epsilon/4$ for all $z \in K$ and $h_x(x) = g_x(x)$. Thus, $h_x(x) = f(x)$ and $h_x(z) < f(z) + \epsilon/2$ for all $z \in K$.

Choose now for each $x \in K$ a function $h_x \in A$ as above. Then, by continuity, for each $x \in K$ there exists an open neighborhood U_x such that $f(z) - \epsilon/2 < h_x(z)$ for all $z \in U_x$. By compactness of K there exists a finite set of points $x_1, \ldots, x_m \in K$ such that the associated neighborhoods $U_{x_1}, \ldots, U_{x_m}$ cover K. Let

$$h := \max\{h_{x_1}, \ldots, h_{x_m}\}.$$

Since K is compact there exists $c > 0$ such that $|h_{x_i}(z)| \leq c$ for all $z \in K$ and all $i \in \{1, \ldots, m\}$. By Lemma 4.5 there exists a polynomial $P_{\max}$ such that $k := P_{\max}(h_{x_1}, \ldots, h_{x_m}) \in A$ satisfies $|k(z) - h(z)| < \epsilon/2$ for all $z \in K$. Then, $f(z) - \epsilon < k(z) < f(z) + \epsilon$ for all $z \in K$. This completes the proof. □

Theorem 4.9 (complex Stone-Weierstraß)**.** *Let K be a compact Hausdorff space and $A \subseteq \mathrm{C}(K, \mathbb{C})$ a subalgebra. Suppose that A separates points, vanishes nowhere and is invariant under complex conjugation. Then, A is dense in $\mathrm{C}(K, \mathbb{C})$ with respect to the topology of uniform convergence.*

Proof. Let $A_{\mathbb{R}}$ be the real subalgebra of A given by the functions with values in $\mathbb{R}$. Note that if $f \in A$, then $\Re f \in A_{\mathbb{R}}$ since $\Re f = (f + \overline{f})/2$. Likewise if $f \in A$, then $\Im f \in A_{\mathbb{R}}$ since $\Im f = -\Re(\mathrm{i} f)$. It is then clear that $A_{\mathbb{R}}$ separates points and vanishes nowhere. Applying the real version of the Stone-Weierstraß Theorem 4.8 we find that $A_{\mathbb{R}}$ is dense in $\mathrm{C}(K, \mathbb{R})$. But then $A = A_{\mathbb{R}} + \mathrm{i} A_{\mathbb{R}}$ is dense in $\mathrm{C}(K, \mathbb{C}) = \mathrm{C}(K, \mathbb{R}) + \mathrm{i}\, \mathrm{C}(K, \mathbb{R})$. □

Theorem 4.10. *Let S be a Hausdorff space and $A \subseteq \mathrm{C}(S, \mathbb{K})$ a subalgebra. Suppose that A separates points, vanishes nowhere and is invariant under complex conjugation if $\mathbb{K} = \mathbb{C}$. Then, A is dense in $\mathrm{C}(S, \mathbb{K})$ with respect to the topology of compact convergence.*

Proof. Recall that the sets of the form

$$U_{K,\epsilon} := \{f \in \mathrm{C}(S, \mathbb{K}) : |f(x)| < \epsilon\ \forall x \in K\},$$

where $K \subseteq S$ is compact and $\epsilon > 0$ form a basis of neighborhoods of 0 in $\mathrm{C}(S, \mathbb{K})$. Given $f \in \mathrm{C}(S, \mathbb{K})$, $K \subseteq S$ compact and $\epsilon > 0$ we have to show that there is $g \in A$ such that $g \in f + U_{K,\epsilon}$. Let A_K be the image of A under the projection $p : \mathrm{C}(S, \mathbb{K}) \to \mathrm{C}(K, \mathbb{K})$. Then, A_K is an algebra that separates points, vanishes nowhere and is invariant under complex conjugation if $\mathbb{K} = \mathbb{C}$. By the ordinary Stone-Weierstraß Theorem 4.8 or 4.9, A_K is dense in $\mathrm{C}(K, \mathbb{K})$ with respect to the topology of uniform convergence. Hence, there exists $g \in A$ such that $p(g) \in B_\epsilon(p(f))$. But this is equivalent to $g \in f + U_{K,\epsilon}$. □

Theorem 4.11. *Let S be a locally compact Hausdorff space and $A \subseteq \mathrm{C}_0(S, \mathbb{K})$ a subalgebra. Suppose that A separates points, vanishes nowhere and is invariant under complex conjugation if $\mathbb{K} = \mathbb{C}$. Then, A is dense in $\mathrm{C}_0(S, \mathbb{K})$ with respect to the topology of uniform convergence.*

Proof. **Exercise.**Hint: Let $\tilde{S} = S \cup \{\infty\}$ be the one-point compactification of S. Show that $\mathrm{C}_0(S, \mathbb{K})$ can be identified with the closed subalgebra $\mathrm{C}_{|\infty=0}(\tilde{S}, \mathbb{K}) \subseteq \mathrm{C}(\tilde{S}, \mathbb{K})$ of those continuous functions on $\tilde{S}$ that vanish at ∞. Denote by $\tilde{A}$ the corresponding extension of A to $\tilde{S}$. Now modify Theorem 4.8 in such a way that $\tilde{A}$ is assumed to vanish nowhere except at ∞ to show that $\tilde{A}$ is dense in $\mathrm{C}_{|\infty=0}(\tilde{S}, \mathbb{K})$. □

4.2 Operators

Definition 4.12. Let X, Y be tvs. We denote the vector space of compact linear maps $X \to Y$ by $\mathrm{KL}(X, Y)$.

Proposition 4.13. *Let X, Y, Z be tvs. Let $f \in \mathrm{CL}(X, Y)$ and $g \in \mathrm{CL}(Y, Z)$. If f or g is bounded, then $g \circ f$ is bounded. If f or g is compact, then $g \circ f$ is compact.*

Proof. **Exercise.** □

Definition 4.14. Let X, Y be normed vector spaces. Then, the *operator norm* on $\mathrm{CL}(X, Y)$ is given by

$$\|f\| := \sup\left\{\|f(x)\| : x \in \overline{B_1(0)} \subseteq X\right\}.$$

Proposition 4.15. *Let X be a normed vector space and Y a Banach space. Then, $\mathrm{CL}(X, Y)$ with the operator norm is a Banach space.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathrm{CL}(X, Y)$. This means,

$$\forall \epsilon > 0 : \exists N > 0 : \forall n, m \geq N : \|f_n - f_m\| \leq \epsilon.$$

But by the definition of the operator norm this is equivalent to

$$\forall \epsilon > 0 : \exists N > 0 : \forall n, m \geq N : \forall x \in X : \|f_n(x) - f_m(x)\| \leq \epsilon\|x\|. \tag{1}$$

Since Y is complete, so each of the Cauchy sequences $\{f_n(x)\}_{n\in\mathbb{N}}$ converges to a vector $f(x) \in Y$. This defines a map $f : X \to Y$. f is linear since we have for all $x, y \in X$ and $\lambda, \mu \in \mathbb{K}$,

$$\begin{aligned} f(\lambda x + \mu y) = \lim_{n\to\infty} f_n(\lambda x + \mu y) &= \lim_{n\to\infty} (\lambda f_n(x) + \mu f_n(y)) \\ &= \lambda \lim_{n\to\infty} f_n(x) + \mu \lim_{n\to\infty} f_n(y) = \lambda f(x) + \mu f(y). \end{aligned}$$

Equation (1) implies now

$$\forall \epsilon > 0 : \exists N > 0 : \forall n \geq N : \forall x \in X : \|f_n(x) - f(x)\| \leq \epsilon \|x\|.$$

This implies that f is continuous and is equivalent to

$$\forall \epsilon > 0 : \exists N > 0 : \forall n \geq N : \|f_n - f\| \leq \epsilon.$$

That is, $\{f_n\}_{n\in\mathbb{N}}$ converges to f. □

Exercise 23. Let X, Y be tvs. Let $\mathfrak{S}$ be the set of bounded subsets of X. (a) Show that $\mathrm{CL}(X, Y)$ is a tvs with the $\mathfrak{S}$-topology. (b) Suppose further that X is locally bounded and Y is complete. Show that then $\mathrm{CL}(X, Y)$ is complete. (c) Show that if X and Y are normed vector spaces the $\mathfrak{S}$-topology coincides with the operator norm topology.

Example 4.16. Let X be a tvs. Then, $\mathrm{CL}(X, X)$ is an algebra over $\mathbb{K}$ and Proposition 4.13 implies that the subsets $\mathrm{BL}(X, X)$ and $\mathrm{KL}(X, X)$ of $\mathrm{CL}(X, X)$ are bi-ideals.

Exercise 24. Let X be a normed vector space. Show that $\mathrm{CL}(X, X)$ with the operator norm and multiplication given by composition is a topological algebra. Moreover, show that $\|A \circ B\| \leq \|A\|\|B\|$ for all $A, B \in \mathrm{CL}(X, X)$.

4.3 Dual spaces

Definition 4.17. Let X be a tvs over $\mathbb{K}$. Then, the space $\mathrm{L}(X, \mathbb{K})$ of linear maps $X \to \mathbb{K}$ is called the *algebraic dual* of X and denoted by $X^\times$. The space $\mathrm{CL}(X, \mathbb{K})$ of continuous linear maps $X \to \mathbb{K}$ is called the *(topological) dual* of X and denoted by X^*.

Definition 4.18. Let X be a tvs. Then, the *weak* topology* on X^* is the coarsest topology on X^* such that the evaluation maps $\hat{x} : X^* \to \mathbb{K}$ given by $\hat{x}(f) := f(x)$ are continuous for all $x \in X$.

Exercise 25. Let X be a tvs. Show that the weak* topology on X^* makes it into a locally convex tvs and indeed coincides with the topology of pointwise convergence under the inclusion $\mathrm{CL}(X, \mathbb{K}) \subseteq \mathrm{C}(X, \mathbb{K})$. Moreover, show that $\mathrm{CL}(X, \mathbb{K})$ is closed in $\mathrm{C}(X, \mathbb{K})$.

Proposition 4.19. *Let X be a tvs, $F \subseteq \mathrm{CL}(X, \mathbb{K})$ equicontinuous. Then, F is bounded in the weak* topology.*

Proof. **Exercise.** □

Proposition 4.20. *Let X be a normed vector space. Then, the operator norm topology on X^* is finer than the weak* topology.*

Proof. **Exercise.** $\square$

Indeed, we shall see that the following Banach-Alaoglu Theorem has as a striking consequence a considerable strengthening of the above statement.

Theorem 4.21 (Banach-Alaoglu)**.** *Let X be a tvs, U a neighborhood of 0 in X and V a bounded and closed set in $\mathbb{K}$. Then, the set*

$$M(U,V) := \{f \in X^* : f(U) \subseteq V\}.$$

is compact with respect to the weak topology.*

Proof. We first show that $M(U,V)$ is closed. To this end observe that

$$M(U,V) = \bigcap_{x\in U} M(\{x\},V) \quad \text{where} \quad M(\{x\},V) := \{f \in X^* : f(x) \in V\}.$$

Each set $M(\{x\},V)$ is closed since it is the preimage of the closed set V under the continuous evaluation map $\hat{x} : X^* \to \mathbb{K}$. Thus, $M(U,V)$, being an intersection of closed sets is closed.

Next we show that $M(U,V)$ is equicontinuous and bounded. Let W be a neighborhood of 0 in $\mathbb{K}$. Since V is bounded there exists $\lambda > 0$ such that $V \subseteq \lambda W$, i.e., $\lambda^{-1}V \subseteq W$. But by linearity $M(U,V) = M(\lambda^{-1}U, \lambda^{-1}V)$. This means that $f(\lambda^{-1}U) \subseteq \lambda^{-1}V \subseteq W$ for all $f \in M(U,V)$, showing equicontinuity. By Proposition 4.19 it is also bounded.

Thus, the assumptions of the Arzela-Ascoli Theorem 3.35 are satisfied and we obtain that $M(U,V)$ is relatively compact with respect to the topology of compact convergence. But since $M(U,V)$ is closed in the topology of pointwise convergence it is also closed in the topology of compact convergence which is finer. Hence, $M(U,V)$ is compact in the topology of compact convergence. But since the topology of pointwise convergence is coarser, $M(U,V)$ must also be compact in this topology. $\square$

Corollary 4.22. *Let X be a normed vector space and $B \subseteq X^*$ the closed unit ball with respect to the operator norm. Then B is compact in the weak* topology.*

Proof. **Exercise.** $\square$

Remark 4.23. Let X be a normed space. Then, X^* with the operator norm topology is complete, i.e., a Banach space (due to Proposition 4.15).

Given a normed vector space X, we shall in the following always equip X^* with the operator norm if not mentioned otherwise.

Definition 4.24. Let X be a normed vector space. The *bidual* space of X, denoted by X^{**} is the dual space of the dual space X^*. Let $x \in X$.

Proposition 4.25. *Let X be a normed vector space. Given $x \in X$ the evaluation map $\hat{x} : X^* \to \mathbb{K}$ given by $\hat{x}(y) := y(x)$ for all $y \in X^*$ is an element of X^{**}. Moreover, the canonical linear map $i_X : X \to X^{**}$ given by $x \mapsto \hat{x}$ is isometric.*

Proof. The continuity of $\hat{x}$ follows from Proposition 4.20. Thus, it is an element of X^{**}. We proceed to show that i_X is isometric. Denote by $\overline{B}_{X^*}$ the closed unit ball in X^*. Then, for all $x \in X$,

$$\|\hat{x}\| = \sup_{f \in \overline{B}_{X^*}} |\hat{x}(f)| = \sup_{f \in \overline{B}_{X^*}} |f(x)| \leq \sup_{f \in \overline{B}_{X^*}} \|f\|\|x\| = \|x\|.$$

On the other hand, given $x \in X$ choose with the help of the Hahn-Banach Theorem (Corollary 3.38) $g \in X^*$ such that $g(x) = \|x\|$ and $\|g\| = 1$. Then,

$$\|\hat{x}\| = \sup_{f \in B_{X^*}} |\hat{x}(f)| \geq |\hat{x}(g)| = |g(x)| = \|x\|.$$

□

Definition 4.26. A Banach space X is called *reflexive* iff the canonical linear map $i_X : X \to X^{**}$ is surjective.

4.4 Adjoint operators

Definition 4.27. Let X, Y be tvs and $f \in \mathrm{CL}(X, Y)$. The adjoint operator $f^* \in \mathrm{L}(Y^*, X^*)$ is defined by

$$(f^*(g))(x) := g(f(x)) \quad \forall x \in X, g \in Y^*.$$

Remark 4.28. It is immediately verified that the image of f^* is indeed contained in X^* and not merely in $X^\times$.

Proposition 4.29. *Let X, Y be tvs and $f \in \mathrm{CL}(X, Y)$. Then, $f^* \in \mathrm{CL}(Y^*, X^*)$ if we equip X^* and Y^* with the weak* topology.*

Proof. **Exercise.** □

Proposition 4.30. *Let X, Y be normed vector spaces and $f \in \mathrm{CL}(X,Y)$. Then, $f^* \in \mathrm{CL}(Y^*, X^*)$ if we equip X^* and Y^* with the operator norm topology. Moreover, equipping also $\mathrm{CL}(X,Y)$ and $\mathrm{CL}(Y^*, X^*)$ with the operator norm we get $\|f^*\| = \|f\|$ for all $f \in \mathrm{CL}(X,Y)$. That is, $^* : \mathrm{CL}(X,Y) \to \mathrm{CL}(Y^*, X^*)$ is a linear isometry.*

Proof. **Exercise.**Hint: Use the Hahn-Banach Theorem in the form of Corollary 3.38 to show that $\|f^*\| \geq \|f\|$. $\square$

Lemma 4.31. *Let X, Y be normed vector spaces and $f \in \mathrm{CL}(X,Y)$. Then, $f^{**} \circ i_X = i_Y \circ f$.*

Proof. **Exercise.** $\square$

Proposition 4.32. *Let X, Y be normed vector spaces and $f \in \mathrm{CL}(X,Y)$. Equip X^* and Y^* with the operator norm topology. Then, f^* is compact iff f is compact.*

Proof. Suppose first that f is compact. Then, $C := \overline{f(B_1(0))}$ is compact. Let B_{Y^*} be the open unit ball in Y^*. Then, B_{Y^*} is equicontinuous and the restriction of B_{Y^*} to $C \subseteq Y$ is bounded in $\mathrm{C}(C, \mathbb{K})$ (with the topology of pointwise convergence). Thus, by the Arzela-Ascoli Theorem 3.35, B_{Y^*} restricted to C is totally bounded in $\mathrm{C}(C, \mathbb{K})$ (with the topology of uniform convergence). In particular, for any $\epsilon > 0$ there exists a finite set $F \subseteq B_{Y^*}$ such that for any $g \in B_{Y^*}$ there is $\tilde{g} \in F$ with $|g(y) - \tilde{g}(y)| < \epsilon$ for all $y \in C$. But then also $|f^*(g)(x) - f^*(\tilde{g})(x)| < \epsilon$ for all $x \in B_1(0) \subseteq X$. This in turn implies $\|f^*(g) - f^*(\tilde{g})\| \leq \epsilon$. That is, $f^*(B_{Y^*})$ is totally bounded and hence relatively compact. Hence, f^* is compact.

Conversely, suppose that f^* is compact. Then, by the same argument as above $f^{**} : X^{**} \to Y^{**}$ is compact. That is, there is a neighborhood U^{**} of 0 in X^{**} such that $f^{**}(U)$ is compact in Y^{**}. Since i_X is continuous $U := i_x^{-1}(U^{**})$ is a neighborhood of 0 in X. Using Lemma 4.31 we get $f^{**}(U^{**}) = f^{**} \circ i_X(U) = i_Y \circ f(U)$. But since i_Y is isometric, the compactness of $i_Y \circ f(U)$ implies the compactness of $f(U)$. Hence, f is compact. $\square$

Proposition 4.33. *Let X, Y be Hausdorff tvs, $A \in \mathrm{CL}(X,Y)$. Then, there are canonical isomorphisms of vector spaces,*

1. $\left(Y/\overline{A(X)}\right)^* \to \ker(A^*)$,
2. $Y^*/\ker(A^*) \to \left(\overline{A(X)}\right)^*$.

Moreover, if we equip dual space with the weak topology, these isomorphisms become isomorphisms of tvs. Similarly, If X and Y are normed vector spaces and we equip dual spaces with the operator norm, the isomorphisms become isometries.*

Proof. Let $q : Y \to Y/\overline{A(X)}$ be the quotient map. The adjoint of q is $q^* : \left(Y/\overline{A(X)}\right)^* \to Y^*$. Since q is surjective, q^* is injective. We claim that the image of q^* is $\ker(A^*) \subseteq Y^*$ proving 1. Let $f \in \left(Y/\overline{A(X)}\right)^*$. Then, $A^*(q^*(f)) = f \circ q \circ A = 0$ since already $q \circ A = 0$. Now suppose $f \in \ker(A^*) \subseteq Y^*$. Then, $f \circ A = 0$, i.e., $f|_{A(X)} = 0$. Since f is continuous, we must actually have $f|_{\overline{A(X)}} = 0$. But this means there is a well defined $g : Y/\overline{A(X)} \to \mathbb{K}$ such that $f = q \circ g$. Moreover, the continuity of f implies continuity of g by the definition of the quotient topology on $Y/\overline{A(X)}$. This completes the proof of 1.

Consider the inclusion $i : \overline{A(X)} \to Y$. The adjoint of i is $i^* : Y^* \to \left(\overline{A(X)}\right)^*$. Since i is injective, i^* is surjective. We claim that the kernel of i^* is precisely $\ker(A^*)$ so that quotienting it leads the isomorphism 2. Indeed, let $f \in Y^*$. $f \in \ker(A^*)$ iff $0 = A^*(f) = f \circ A$. But this is equivalent to $f|_{A(X)} = 0$. Since f is continuous this is in turn equivalent to $f|_{\overline{A(X)}} = 0$. But this is in turn equivalent to $0 = f \circ i = i^*(f)$, completing the proof of 2.

Exercise. Complete the topological part of the proof. □

4.5 Approximating Compact Operators

Definition 4.34. Let X, Y be tvs. We denote the space of continuous linear maps $X \to Y$ with finite dimensional image by $\mathrm{CL}_{\mathrm{fin}}(X, Y)$.

Proposition 4.35. *Let X, Y be tvs such that Y is Hausdorff. Then,* $\mathrm{CL}_{\mathit{fin}}(X, Y) \subseteq \mathrm{KL}(X, Y)$.

Proof. **Exercise.** □

Proposition 4.36. *Let X, Y be normed vector spaces. Then,* $\overline{\mathrm{CL}_{\mathit{fin}}(X, Y)} \subseteq \mathrm{KL}(X, Y)$ *with respect to the operator norm topology.*

Proof. Let $f \in \overline{\mathrm{CL}_{\mathrm{fin}}(X, Y)}$ and $\epsilon > 0$. Then, there exists $g \in \mathrm{CL}_{\mathrm{fin}}(X, Y)$ such that $\|f - g\| < \epsilon$. In particular, $(f - g)(\overline{B_1(0)}) \subseteq B_\epsilon(0)$. This implies $f(\overline{B_1(0)}) \subseteq g(\overline{B_1(0)}) + B_\epsilon(0)$. But $g(\overline{B_1(0)})$ is a bounded subset of the finite dimensional subspace $g(X)$ and hence totally bounded. Thus, there exists a finite subset $F \subseteq g(\overline{B_1(0)})$ such that $g(\overline{B_1(0)}) \subseteq F + B_\epsilon(0)$. But then,

$f(\overline{B_1(0)}) \subseteq F + B_\epsilon(0) + B_\epsilon(0) \subseteq F + B_{2\epsilon}(0)$. That is, $f(\overline{B_1(0)})$ is covered by a finite number of balls of radius 2ϵ. Since ϵ was arbitrary this means that $f(\overline{B_1(0)})$ is totally bounded and hence relatively compact. □

Proposition 4.37. *Let X, Y be normed vector spaces. Suppose there exists a bounded sequence $\{s_n\}_{n\in\mathbb{N}}$ of operators $s_n \in \mathrm{CL}_{fin}(Y,Y)$ such that $\lim_{n\to\infty} s_n(y) = y$ for all $y \in Y$. Then, $\mathrm{KL}(X,Y) \subseteq \overline{\mathrm{CL}_{fin}(X,Y)}$ with respect to the operator norm topology.*

Proof. **Exercise.**Hint: For $f \in \mathrm{KL}(X,Y)$ and $\epsilon > 0$ show that there exists $n \in \mathbb{N}$ such that $\|s_n \circ f - f\| < \epsilon$. □

4.6 Fredholm Operators

Proposition 4.38. *Let X be a Hausdorff tvs and $T \in \mathrm{KL}(X,X)$. Then, the kernel of $S := \mathbf{1} - T \in \mathrm{CL}(X,X)$ is finite-dimensional.*

Proof. Note that T acts as the identity on the subspace $\ker S$. Denote this induced operator by $\tilde{T} : \ker S \to \ker S$. Since T is compact so is $\tilde{T}$. Thus, there exists a neighborhood of 0 in $\ker S$ that is compact. In particular, $\ker S$ is locally compact. By Theorem 3.18, $\ker S$ is finite dimensional. □

Proposition 4.39. *Let X, Y be Banach spaces and $f \in \mathrm{CL}(X,Y)$ injective. Then, $f(X)$ is closed iff there exists $c > 0$ such that $\|f(x)\| \geq c\|x\|$ for all $x \in X$.*

Proof. Suppose first that $f(X)$ is closed. Then, $f(X)$ is complete since Y is complete. Thus, by Corollary 3.66, f is open as a map $X \to f(X)$. In particular, $f(B_1(0))$ is an open neighborhood of 0 in $f(X)$. Thus, there exists $c > 0$ such that $B_c(0) \subseteq f(B_1(0)) \subseteq f(X)$. By injectivity of f this implies that $\|f(x)\| \geq c$ for all $x \in X$ with $\|x\| \geq 1$. This implies in turn $\|f(x)\| \geq c\|x\|$ for all $x \in X$.

Conversely, assume that there is $c > 0$ such that $\|f(x)\| \geq c\|x\|$ for all $x \in X$. Let $y \in \overline{f(X)}$. Then there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ in X such that $\{f(x_n)\}_{n\in\mathbb{N}}$ converges to y. In particular, $\{f(x_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence. But as is easy to see the assumption then implies that $\{x_n\}_{n\in\mathbb{N}}$ is also a Cauchy sequence. Since X is complete this sequence converges, say to $x \in X$. But since f is continuous we must have

$$y = \lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right) = f(x).$$

In particular, $y \in f(X)$, i.e., $f(X)$ is closed. □

Proposition 4.40. *Let X be a Banach space and $T \in \mathrm{KL}(X, X)$. Then, the image of $S := \mathbf{1} - T \in \mathrm{CL}(X, X)$ is closed and has finite codimension, i.e., $X/S(X)$ is finite dimensional.*

Proof. We first show that $S(X)$ is a closed subspace of X. Since S is continuous $\ker S$ is a closed subspace of X. The quotient map $q : X \to X/\ker(S)$ is thus a continuous and open linear map between Banach spaces. S factorizes through q via $S = \tilde{S} \circ q$, where $\tilde{S} : X/\ker(S) \to X$ is linear, continuous and injective. We equip $X/\ker(S)$ with the quotient norm. By Theorem 3.64 this space is a Banach space. By Proposition 4.39 the image of $\tilde{S}$ (and thus that of S) is closed iff there exists a constant $c > 0$ such that $\|\tilde{S}(y)\| \geq c\|y\|$ for all $y \in S/\ker(S)$. Hence, we have to demonstrate the existence of such a constant. Suppose it does not exist. Then, there is a sequence $\{y_n\}_{n\in\mathbb{N}}$ of elements of $X/\ker(S)$ with $\|y_n\| = 1$ and such that $\lim_{n\to\infty} \tilde{S}(y_n) = 0$. Now choose a preimages x_n of the y_n in X with $1 \leq \|x_n\| < 2$. Then, $\{x_n\}_{n\in\mathbb{N}}$ is bounded so that $\{T(x_n)\}_{n\in\mathbb{N}}$ is compact. In particular, there is a subsequence $\{x_k\}_{k\in\mathbb{N}}$ so that $\{T(x_k)\}_{k\in\mathbb{N}}$ converges, say to $z \in X$. Since on the other hand $\lim_{k\to\infty} S(x_k) = 0$ we find with $S + T = \mathbf{1}$ that $\lim_{k\to\infty} x_k = z$. So by continuity of S we get $S(z) = 0$, i.e., $z \in \ker(S)$ and hence $z \in \ker q$. By continuity of q this implies, $\lim_{k\to\infty} \|q(x_k)\| = 0$, contradicting $\|q(x_k)\| = \|y_k\| = 1$ for all $k \in \mathbb{N}$. This completes the proof of the existence of c and hence of the closedness of the image of S.

The compactness of T implies the compactness of T^* by Proposition 4.32. Thus, by Proposition 4.38, $S^* = \mathbf{1}^* - T^*$ has finite dimensional kernel. But Proposition 4.33.1 implies then that the codimension of $\overline{S(X)}$ in X, i.e., the dimension of $X/\overline{S(X)}$ is also finite. Since we have seen above that $\overline{S(X)} = S(X)$, this completes the proof. □

Definition 4.41. Let X, Y be normed vector spaces and $A \in \mathrm{CL}(X, Y)$. A is called a *Fredholm operator* iff the kernel of A is finite dimensional and its image is closed and of finite codimension. Then, we define the *index* of a A to be

$$\operatorname{ind} A = \dim(\ker A) - \dim(Y/A(Y)).$$

We denote by $\mathrm{FL}(X, Y)$ the set of Fredholm operators.

Proposition 4.42. *Let X be a Hausdorff tvs. Then, any finite dimensional subspace of X is closed.*

Proof. Let $A \subseteq X$ be a subspace of dimension n. Then, A as a tvs is isomorphic to $\mathbb{K}^n$. In particular, A is complete and thus closed in X by Proposition 3.15. □

Proposition 4.43. *Let X be a Hausdorff tvs, C a closed subspace of X and F a finite-dimensional subspace of X. Then, $F + C$ is closed in X.*

Proof. Since C is closed X/C is a Hausdorff tvs. Let $p : X \to X/C$ be the continuous projection. Then, $p(F)$ is finite-dimensional, hence complete and closed in X/C. Thus, $F + C = p^{-1}(p(F))$ is closed. □

Proposition 4.44. *Let X be a locally convex Hausdorff tvs. Then, any finite dimensional subspace of X admits a closed complement.*

Proof. We proceed by induction in dimension. Let $A \subseteq X$ be a subspace of dimension 1 and $v \in A \setminus \{0\}$. Define the linear map $\lambda : A \to \mathbb{K}$ by $\lambda(v) = 1$. Then, the Hahn-Banach Theorem in the form of Theorem 3.39 ensures that λ extends to a continuous map $\tilde{\lambda} : X \to \mathbb{K}$. Then, clearly $\ker \tilde{\lambda}$ is a closed complement of A in X. Now suppose we have shown that for any subspace of dimension n a closed complement exists in X. Let N be a subspace of X of dimension $n + 1$. Choose an n-dimensional subspace $M \subset N$. This has a closed complement C by assumption. Moreover, C is a locally convex Hausdorff tvs in its own right. Let $A = N \cap C$. Then, A is a one-dimensional subspace of C and we can apply the initial part of the proof to conclude that it has a closed complement D in C. But D is closed also in X since C is closed in X and it is a complement of N. □

Lemma 4.45 (Riesz)**.** *Let X be a normed vector space and C a closed subspace. Then, for any $1 > \epsilon > 0$ there exists $x \in X \setminus C$ with $\|x\| = 1$ such that for all $y \in C$,*

$$\|x - y\| \geq 1 - \epsilon.$$

Proof. Choose $x_0 \in X \setminus C$ arbitrary. Now choose $y_0 \in C$ such that

$$\|x_0 - y_0\| \leq \|x_0 - y\| \frac{1}{1 - \epsilon}$$

for all $y \in C$. We claim that

$$x := \frac{x_0 - y_0}{\|x_0 - y_0\|}$$

has the desired property. Indeed, for all $y \in C$,

$$\|x - y\| = \frac{\|x_0 - y_0 - (\|x_0 - y_0\|)y\|}{\|x_0 - y_0\|} \geq \frac{\|x_0 - y_0\|(1 - \epsilon)}{\|x_0 - y_0\|}.$$

□

Proposition 4.46. *Let X, Y be Banach spaces. Then,* $\mathrm{FL}(X,Y)$ *is open in* $\mathrm{CL}(X,Y)$*. Moreover,* $\mathrm{ind} : \mathrm{FL}(X,Y) \to \mathbb{Z}$ *is continuous.*

Proof. Let $S : X \to Y$ be Fredholm. Since $\ker S$ is finite dimensional, there exists a closed complement $C \subseteq X$ by Proposition 4.44. Then, $S|_C : C \to Y$ is injective and has closed image $S(C) = S(X)$. Thus, by Proposition 4.39 there exists $c > 0$ such that $\|S(x)\| \geq c\|x\|$ for all $x \in C$. Now consider $T \in B_{c/2}(S) \subseteq \mathrm{CL}(X,Y)$. We claim that T is Fredholm and that $\mathrm{ind}\, T = \mathrm{ind}\, S$, thus proving the assertions. Indeed, for all $x \in C$ we have

$$\|T(x)\| \geq \|S(x)\| - \|S(x) - T(x)\| \geq c\|x\| - \|S - T\|\|x\| \geq c/2\|x\|.$$

Thus, $\ker T \cap C = \{0\}$ and so the dimension of $\ker T$ must be smaller or equal to the codimension of C, which is finite. Also, $T|_C$ is injective and has closed image by Proposition 4.39. But $T(X) = T(C) + T(\ker S)$, so by Proposition 4.43, the image of T is closed. We proceed to show that $S(C) \subseteq T(C)$. Assume the contrary. Then, by Lemma 4.45 there exists $y \in S(C) \setminus T(C)$ with $\|y\| = 1$ such that $\|y - z\| \geq 1/2$ for all $z \in T(C) \cap S(C)$. Let $x := (S|_C)^{-1}(y)$. Then, $\|x\| \leq 1/c$ and we have $\|S(x) - T(x)\| \geq 1/2$. But,

$$1/2 > \|S - T\| \frac{1}{c} \geq \|S - T\|\|x\| \geq \|S(x) - T(x)\|,$$

yielding a contradiction and proving that $S(C) \subseteq T(C)$. This implies in particular, that $T(X)$ has finite codimension and completes the proof that T is Fredholm.

Note that the same argument as above also yields $T(C) \subseteq S(C)$ and hence $T(C) = S(C)$. Since $\ker T \cap C = \{0\}$, there exists a subspace $N \subseteq X$ such that $X = C \oplus N \oplus \ker T$ as tvs. (Note that N is finite-dimensional because $\dim N = \dim(\ker S) - \dim(\ker T)$ and hence closed.) Then, $T(X) = T(C + N)$. But T is injective on $C \oplus N$. So,

$$\begin{aligned}
&\dim(Y/T(X)) = \dim(Y/T(C+N)) = \dim(Y/T(C)) - \dim T(N) \\
&\quad = \dim(Y/S(C)) - \dim N = \dim(Y/S(X)) - \dim(\ker S) + \dim(\ker T).
\end{aligned}$$

In particular, this implies $\mathrm{ind}\, T = \mathrm{ind}\, S$. □

Corollary 4.47. *Let X be a Banach space and $T \in \mathrm{KL}(X,X)$. Then,* $S := \mathbf{1} - T \in \mathrm{FL}(X,X)$*. Moreover,* $\mathrm{ind}\, S = 0$*.*

Proof. **Exercise.**Hint: For the second assertion consider the family of operators $S_t := \mathbf{1} - tT$ for $t \in [0,1]$ and use the continuity of ind. □

Proposition 4.48 (Fredholm alternative). *Let X be a Banach spaces, $T \in \mathrm{KL}(X,X)$ and $\lambda \in \mathbb{K} \setminus \{0\}$. Then, either the equation*

$$\lambda x - Tx = y$$

has one unique solution $x \in X$ for each $y \in X$, or it has no solution for some $y \in X$ and infinitely many solutions for all other $y \in X$.

Proof. **Exercise.** □

4.7 Eigenvalues and Eigenvectors

Definition 4.49. Let X be a tvs and $A \in \mathrm{CL}(X,X)$. Then, $\lambda \in \mathbb{K}$ is called an *eigenvalue* of A iff there exists $x \in X \setminus \{0\}$ such that $\lambda x - Ax = 0$. Then x is called an *eigenvector* for the eigenvalue λ. Moreover, the vector space of eigenvectors for the eigenvalue λ is called the *eigenspace* of λ.

Proposition 4.50. *Let X be a Banach space and $T \in \mathrm{KL}(X,X)$. Then, $\lambda \in \mathbb{K} \setminus \{0\}$ is an eigenvalue of T iff $\lambda\mathbf{1} - T$ does not have a continuous inverse.*

Proof. **Exercise.** □

Lemma 4.51. *Let X be a Banach space, $T \in \mathrm{KL}(X,X)$ and $c > 0$. Then, the set of eigenvalues λ such that $|\lambda| > c$ is finite.*

Proof. Suppose the assertion is not true. Thus, there exists a sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ of distinct eigenvalues of T such that $|\lambda_n| > c$ for all $n \in \mathbb{N}$. Let $\{v_n\}_{n\in\mathbb{N}}$ be a sequence of associated eigenvectors. Observe that the set of these eigenvectors is linearly independent. For all $n \in \mathbb{N}$ let A_n be the vector space spanned by $\{v_1, \dots, v_n\}$. Thus $\{A_n\}_{n\in\mathbb{N}}$ is a strictly ascending sequence of finite-dimensional subspaces of X. Set $y_1 := v_1/\|v_1\|$. Using Lemma 4.45 we choose for each $n \in \mathbb{N}$ a vector $y_{n+1} \in A_{n+1}$ such that $\|y_{n+1}\| = 1$ and $\|y_{n+1} - y\| > 1/2$ for all $y \in A_n$. Now let $n > m \geq 1$. Then,

$$\begin{aligned}\|Ty_n - Ty_m\| &= \|\lambda_n y_n - (\lambda_n y_n - Ty_n + Ty_m)\| \\ &= |\lambda_n| \|y_n - |\lambda_n|^{-1}(\lambda_n y_n - Ty_n + Ty_m)\| > |\lambda_n|\frac{1}{2} > \frac{1}{2}c.\end{aligned}$$

We have used here that $\lambda_n y_n - Ty_n \in A_{n-1}$ and that $Ty_m \in A_m \subseteq A_{n-1}$. This shows that the image of the bounded set $\{y_n\}_{n\in\mathbb{N}}$ under T is not totally bounded. But this contradicts the compactness of T. □

Definition 4.52. Let X be a Banach space and $A \in \mathrm{CL}(X, X)$. Then, the set $\sigma(A) := \{\lambda \in \mathbb{K} : \lambda\mathbf{1} - A$ is not continuously invertible$\}$ is called the *spectrum* of A.

Theorem 4.53. *Let X be a Banach space and $T \in \mathrm{KL}(X, X)$.*

1. *If X is infinite-dimensional, then $0 \in \sigma(T)$.*
2. *The set $\sigma(T)$ is bounded.*
3. *The set $\sigma(T)$ is countable.*
4. *$\sigma(T)$ has at most one accumulation point, 0.*

Proof. **Exercise.** □